

Market Price Manipulation in a Sequential Trade Model*

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Abstract. The dynamic version of the Glosten and Milgrom (1985) model of asset pricing with asymmetric information is studied. It is shown that there is a unique equilibrium when the next-period value function of the informed trader, who knows the terminal value of the asset, is strictly convex and strictly monotone in terms of the market maker's prior belief. A characterization of the bid and ask prices and the informed trader's manipulative strategy in equilibrium is given. Finally, a computational method for simulating the equilibrium is presented.

Key Words: Market microstructure; Glosten-Milgrom; Dynamic trading; Price formation; Sequential trade; Asymmetric information; Bid-ask spreads.

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1 Introduction

This paper develops a model of dynamic informed trading from a canonical framework in the market microstructure literature and characterizes an equilibrium. In asymmetric information models of financial markets, trading behavior imperfectly reveals the private information held by traders. Informed traders who trade dynamically thus have an incentive not only to trade less aggressively but also to manipulate the market by trading in the wrong direction, undertaking short-term losses to confuse the market and then recouping the losses in the future. Dynamic trading and price manipulation by an informed trader have been a challenging issue in the literature of market microstructure.

There are two standard reference frameworks in the literature. The first is called the “continuous auction framework” first developed by Kyle (1985). The second is the “sequential trade framework” proposed by Glosten and Milgrom (1985). A large amount of research has been done involving the application of these two frameworks. Both frameworks are sufficiently simple and well behaved that they easily lend themselves to analysis of policy issues and empirical testing (see Madhavan (2000) and Biais et al. (2005) for extensive surveys of the literature).

One of the simplifying assumptions in Glosten and Milgrom (1985) is that traders can trade only once. In the original Glosten-Milgrom model manipulation does not occur because there is no chance to re-trade and as a result traders maximize their one-period payoff. In the Kyle model the informed trader’s strategy is monotonic in the sense that she buys the asset when it is undervalued given her information and vice versa; dynamic price manipulation is ruled out by assumption.

This paper follows the strand of the sequential trade framework developed by Glosten and Milgrom. This paper considers markets where a risky asset is traded for finitely many periods between competitive market makers, two types of strategic informed traders and liquidity traders. In the beginning of the game, nature chooses the liquidation value of a risky asset to be high or low and tells the informed trader who trades dynamically. In each period there is a random determination of whether the informed trader or a liquidity trader trades. The market maker posts bid and ask prices for the next period, after which the trader buys or sells one unit. The termination value is revealed at the end of the game and the payoff for the informed trader is the sum of the termination value times net-holding of the asset and revenue from buying and selling the asset. Within the model described above we consider an equilibrium such that (a) the informed trader’s strategy is optimal beginning at any history; (b) market makers make zero profit in each period under their common Bayesian belief conditional on the history and chosen trade; (c) liquidity traders trade for their exogenous liquidity needs.

There exist a large number of papers that theoretically and empirically examine price manipulation in different settings. For example, Jordan and Jordan (1996) examine Solomon Brothers’ market corner of a Treasury note auction in May 1991. Felixson and Pelli (1999) examine closing price manipulation in the Finnish stock market. Mahoney (1999) examines stock price manipulation leading up to the Securities Exchange Act of 1934. Vitale (2000) examines manipulation in the foreign exchange market. Merrick et al. (2005) examine a case of manipulation involving a delivery squeeze on a bond futures

contract traded in London. A recent paper by Aggarwal and Wu (2006) suggests that stock market manipulation may have important impacts on market efficiency. According to the empirical findings in Aggarwal and Wu (2006), while manipulative activities seem to have declined on the main exchanges, it is still a serious issue in both developed and emerging financial markets, especially in the over-the-counter markets.

The theoretical literature starts with manipulation by uninformed traders. Allen and Gale (1992) provide a model of strategic trading in which some equilibria involve manipulation. Furthermore, Allen and Gorton (1992) consider a model of pure trade-based uninformed manipulation in which an asymmetry in buys and sells by liquidity traders creates the possibility of manipulation. The first paper to consider manipulation by an informed trader within the discrete-time Glosten-Milgrom framework is Chakraborty and Yilmaz (2004). They show that when the market faces uncertainty about the existence of informed traders and there are a large number of trading periods long-lived informed traders will manipulate in every equilibrium. On the other hand, Back and Baruch (2004) study the equivalence of the Glosten-Milgrom model and the Kyle model in a continuous-time setting, and show that the equilibrium in the Glosten-Milgrom model is approximately the same as that in the Kyle model when the trade size is small and uninformed trades arrive frequently. They conclude that the continuous-time Kyle model is more tractable than the Glosten-Milgrom model, although most markets are organized as in the sequential trade models. More recently there has been an interest in the informed trader's dynamic strategy. Among others, Brunnermeier and Pedersen (2005) consider dynamic strategic behavior of large traders and show that "overshooting" occurs in equilibrium. Back and Baruch (2007) analyze different market systems by allowing the informed traders to trade continuously within the Glosten-Milgrom framework.

Despite the importance of dynamic trading strategies by informed traders in the literature, characteristics of price dynamics and information transmission have not yet been adequately studied because there is no closed-form solution for equilibrium in the dynamic Glosten-Milgrom framework and it is not yet known if equilibrium is unique either in the Kyle model (see Boulatov et al. (2005) for a further development) or the dynamic Glosten-Milgrom model in which strategic informed traders can trade repeatedly. In this paper we present a model of dynamic informed trading and show that there exists a unique equilibrium when the value functions of the informed traders are monotonic and strictly convex in terms of the market maker's prior belief. In addition, we characterize the equilibrium bid and ask prices and specify the necessary condition for manipulation to occur in equilibrium. Finally, we present a computational method to solve for the equilibrium and comparative statics from the simulations.

It is worth mentioning that in Back and Baruch (2004) the equilibrium strategy of the informed trader is not accurately simulated. This is because their model is a continuous-time stationary case and their program tries to find the value functions as a fixed point. To do this they use an extrapolation method which requires calculating the slopes of the value functions. In the region of beliefs at which manipulation arises the slopes of the value functions are very small. In fact, Back and Baruch (2004)

wrote that even though all the equilibrium conditions hold with a high order of accuracy, it appears from their plots that the strategies are not estimated very accurately when manipulation arises and this is probably an inevitable result of their estimation method, because the derivative of the value function is small where manipulation arises (see Back and Baruch (2004) p.464, last paragraph).

This difficulty prevents accurate analysis of the informed trader’s manipulative strategy even if the value functions are known exactly. The method developed in this paper takes advantage of the deterministic construction of our model and allows us to accurately analyze the informed trader’s manipulative strategy. Instead of using the extrapolation method we find bid and ask prices to make the informed trader indifferent between the two orders: buy and sell. In this way we can solve for the equilibrium directly. This method is one of the significant contributions that we make to the literature.

The paper is organized as follows. The second section presents the model. The third section proves the uniqueness of equilibrium under specific assumptions for the value functions of the informed traders. The fourth section characterizes an equilibrium and provides a numerical simulation of the model. The fifth section concludes.

2 The model

Trade occurs for finitely many periods, denoted by $t = 1, 2, \dots, T$. Each interval of time accommodates one trade. There is a risky asset and a numeraire in terms of which the asset price is quoted. The terminal value of the risky asset, denoted by \tilde{v} , is a random variable which can take the value 0 or 1. The risk-free interest rate is assumed to be zero.

There are two kinds of orders available to traders: sell or buy. Let $A = \{S, B\}$ where S denotes a sell order and B denotes a buy order. Let $\Delta(A)$ denote the set of probability distributions on A . Let h_t denote the order that the market maker receives in period t , i.e. h_t is the realized order in period t .

There are three classes of risk-neutral market participants: competitive market makers, an informed trader and a liquidity trader. Trade arises from both informed traders, who know the terminal value of the asset, and uninformed traders. The type of the trader arriving in period t is determined by a random variable $\tilde{\tau}_t$, which takes values from the set $\{i, l\}$. The letters i and l respectively denote the informed type and the liquidity type. The random variables $\{\tilde{\tau}_t : t = 1, \dots, T\}$ are i.i.d. across the periods $1, \dots, T$ and satisfy $\Pr(\tilde{\tau}_t = i) = \mu$. If the trader’s type in period t is l , then the demand in that period is determined by the random variable \tilde{Q}_t which takes values from A . The random variables $\{\tilde{Q}_t : t = 1, \dots, T\}$ are i.i.d. and satisfy $\Pr(\tilde{Q}_t = B) = \gamma > 0$. For any given period t , the random variables $\tilde{\tau}_t, \tilde{Q}_t, \tilde{v}$ are mutually independent.

The private information of the informed trader is determined by a random variable $\tilde{\theta} \in \Theta = \{H, L\}$. When $\theta = L$, the informed knows that the value of the asset is 0. We call this type of trader “low-type” and denote him by L . When $\theta = H$, the informed trader knows that the value of the asset is 1. We call this type of trader “high-type” and denote him by H . Only one type of trader is actually

chosen by nature to trade for any given play of the game.

Next we describe the details with regard to the market maker's pricing strategy and the informed trader's trading strategy. First, we define the space of all possible trading orders. When the traders choose their orders and the market maker posts the bid and ask prices in period t , each knows the entire trading history until and including period $t - 1$. A period- t history $h^t := (h_1, \dots, h_t)$ is a sequence of realized orders for periods until and including $t - 1$. Let $\mathcal{H}^t := \underbrace{A \times \dots \times A}_{t \text{ times}}$. The space of all possible period- t histories, $t \geq 1$, is described by $\mathcal{H} = \cup_{t=1}^T \mathcal{H}^t$. A history h^t is taken to be the generic element of \mathcal{H} . For notational convenience we let $h^0 = \emptyset$.

Knowledge of the game structure and of the parameters of the joint distribution of the traders state variables is common to all market participants. In each period market makers post bid and ask prices equal to the expected value of the asset conditional on the observed history of trades. The trader trades at those prices. Trading happens for finitely many periods after which all private information is revealed.

The timing structure of the trading game is as follows:

1. In period 0 nature chooses the realization $v \in \{0, 1\}$ of the risky asset payoff \tilde{v} and the type of the informed trader $\theta \in \{H, L\}$. The informed trader observes θ .
2. In successive periods, indexed by $t = 1, \dots, T$, having observed the realized trades in periods $1, \dots, t - 1$, the competitive market maker posts bid and ask prices. Nature chooses an informed trader of type θ with probability μ and a liquidity trader with probability $1 - \mu$. The trader learns market maker's price quote.
3. If the trader is informed he takes the profit-maximizing quote. If the trader is a liquidity trader he trades according to his liquidity needs.
4. In period T , the realization of v is publicly disclosed.

A price rule, specifying bid and ask prices that will be posted by the market makers in the beginning of period t , is defined as a function $p_t : \mathcal{H} \rightarrow [0, 1]^2$ with $p_t = (\beta_t, \alpha_t)$. For each trader a trading strategy specifies a probability distribution over trades in period t with respect to the bid and ask prices p_t posted in period t . A strategy for an informed trader is defined as a function $\sigma_\theta : \mathcal{H} \rightarrow \Delta(A)$. For each $\theta \in \Theta = \{H, L\}$ and $a \in A = \{B, S\}$, $\sigma_{\theta a}(h^t)$ is the probability that σ_θ assigns to action a after history h^t . That is, $\sigma_{HS}(h^t)$ denotes the probability that the high-type assigns to selling conditional on history h^t .

To determine bid and ask prices to be posted in period t the market maker updates his prior conditional on the arrival of an order of the relevant type. Let $\delta : \mathcal{H} \rightarrow \Delta(\{0, 1\})$ be the market maker's prior belief at the beginning of period t that the risky asset's value is high conditional on history h^{t-1} . The belief is updated through Bayes' rule.

Definition 1. A high-type informed trader's strategy is optimal after history h^{t-1} in response to prices $p_t = (\alpha_t, \beta_t)$ if it prescribes a probability distribution $\sigma_H^* \in \Delta(A)$ over $a \in A$ such that for every t and $h^t \in \mathcal{H}$,

$$\sigma_H^*(h^{t-1}) \in \arg \max_{\sigma_H \in \Delta(A)} \sum_{s=t}^T \mu [\sigma_{HB}[1 - \alpha_s(h^{s-1})] - \sigma_{HS}[1 - \beta_s(h^{s-1})]]. \quad (1)$$

Definition 2. Similarly, a low-type informed trader's strategy is optimal after history h^{t-1} in response to $p_t = (\alpha_t, \beta_t)$ if it prescribes a probability distribution $\sigma_L^* \in \Delta(A)$ over $a \in A$ such that for every t and $h^t \in \mathcal{H}$,

$$\sigma_L^*(h^{t-1}) \in \arg \max_{\sigma_L \in \Delta(A)} \sum_{s=t}^T \mu [-\sigma_{LB}\alpha_s(h^{s-1}) + \sigma_{LS}\beta_s(h^{s-1})]. \quad (2)$$

Next we define an equilibrium for our economy:

Definition 3. An equilibrium consists of a pair of bid and ask prices $\{p_t^* = (\beta_t^*, \alpha_t^*)\}_{t \in \{1, \dots, T\}}$, and an informed trader's strategies $\sigma^* = (\sigma_L^*, \sigma_H^*)$ such that for all $t \in \{1, \dots, T\}$ and for all $h^{t-1} \in \mathcal{H}$,

(P1) the pair of bid and ask prices p_t^* satisfies the zero-profit condition with respect to the market maker's posterior belief: $\alpha_t^*(h^{t-1}) = E[v|h^{t-1}, h_t = B]$, and $\beta_t^*(h^{t-1}) = E[v|h^{t-1}, h_t = S]$;

(P2) the informed trader's strategies σ_H^* and σ_L^* are optimal given the pair of bid and ask prices p_t^* ;

(B) the pair of bid and ask prices $p_t^* = (\beta_t^*, \alpha_t^*)$ satisfies Bayes' rule.

Now, we define a manipulative strategy. We say that a strategy is manipulative if it involves the informed trader undertaking a trade in any period that yields a strictly negative short-term profit. If this occurs in equilibrium it means that manipulation enables the informed trader to recoup the short-term losses.

Definition 4. Given a pair of bid and ask prices p_t for some $t \in \{1, \dots, T\}$ and a history $h^{t-1} \in \mathcal{H}$, a strategy σ_θ is called manipulative in period t for the high type if $\sigma_{HS}(h^{t-1}) > 0$; or for the low type if $\sigma_{LB}(h^{t-1}) > 0$.

This is the same definition used by Chakraborty and Yilmaz (2004). Back and Baruch (2004) used the term "bluffing" instead. We call the situation where the informed trader chooses a totally mixed strategy "price manipulation." It's worth mentioning that in Huberman and Stanzl (2004) a price manipulation is defined as a round-trip trade. In this paper price manipulation occurs as a round-trip trade in equilibrium but not by definition. This is because if the informed trader trades against his short term profit incentive he incurs a loss which must be recouped, consequently price manipulation takes the form of a round-trip trade in equilibrium. We now prove the existence of an equilibrium.

Theorem 1. An equilibrium exists.

Proof: Found in the Appendix. ■

3 Uniqueness of equilibrium

Fix a history h^t arbitrarily and suppose that $b = \delta(h^t)$. In this section we will prove the uniqueness of equilibrium if the next-period value function is strictly monotonic and convex in the market maker's belief b . Let W_H and W_L represent the current value of the game for both traders. Let V_H and V_L represent the continuation value of the remainder of the game for both traders. The values of the game are defined as:

$$W_H(b, \sigma) = \mu (\sigma_{HB}(1 - \alpha(b, \sigma) + V_H(\alpha(b, \sigma))) + \sigma_{HS}(\beta(b, \sigma) - 1 + V_H(\beta(b, \sigma)))) \\ + (1 - \mu) (\gamma V_H(\alpha(b, \sigma)) + (1 - \gamma)V_H(\beta(b, \sigma))),$$

and

$$W_L(b, \sigma) = \mu (\sigma_{LB}(-\alpha(b, \sigma) + V_L(\alpha(b, \sigma))) + \sigma_{LS}(\beta(b, \sigma) + V_L(\beta(b, \sigma)))) \\ + (1 - \mu) (\gamma V_L(\alpha(b, \sigma)) + (1 - \gamma)V_L(\beta(b, \sigma))).$$

Throughout the next two sections we assume that the value functions are functions of the market maker's belief and that they satisfy the following two conditions:

- (C1) V_H is monotonically decreasing and V_L is monotonically increasing in terms of the market maker's belief;
- (C2) V_H and V_L are strictly convex in terms of the market maker's belief.

To show that an equilibrium exists uniquely we first show that under these conditions the bid-ask spread is always strictly positive. In this sense, there is no pure arbitrage opportunity for the informed traders. In what follows the proofs are kept in the Appendix unless otherwise specified.

Lemma 1. *In equilibrium, the followings hold:*

1. $\alpha(b, \sigma) < b < \beta(b, \sigma)$;
2. $\sigma_{HB} > \sigma_{LB}$ and $\sigma_{HS} < \sigma_{LS}$.

In equilibrium the high-type trader will not sell with probability *one* and the low-type trader will not buy with probability *one*. This means that an informed trader either trades on his information or assigns a positive probability to both buy and sell orders. In the latter case the informed trader is indifferent between buy and sell orders. This motivates the following lemma.

Lemma 2. *Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b . Then, the following holds:*

$$W_H(b, \sigma) = \mu (1 - \alpha(b, \sigma) + V_H(\alpha(b, \sigma))) + (1 - \mu) (\gamma V_H(\alpha(b, \sigma)) + (1 - \gamma)V_H(\beta(b, \sigma))), \quad (3)$$

and

$$W_L(b, \sigma) = \mu (\beta(b, \sigma) + V_L(\beta(b, \sigma))) + (1 - \mu) (\gamma V_L(\alpha(b, \sigma)) + (1 - \gamma)V_L(\beta(b, \sigma))). \quad (4)$$

Proof of Lemma 2: Omitted. ■

Next, we consider the slopes of the value functions. If the low-type manipulates we have:

$$d_L(b, \sigma) \equiv \frac{V_L(\alpha(b, \sigma)) - V_L(\beta(b, \sigma))}{\alpha(b, \sigma) - \beta(b, \sigma)} = \frac{\alpha(b, \sigma) + \beta(b, \sigma)}{\alpha(b, \sigma) - \beta(b, \sigma)} = 1 + \frac{2\beta(b, \sigma)}{\alpha(b, \sigma) - \beta(b, \sigma)}. \quad (5)$$

This means that if the low-type manipulates, then the average slope between the ask and bid price in the value function is greater than 1. A similar argument also holds for the high-type. Thus we conclude the following.

Lemma 3. L. *If the low-type takes a manipulative strategy at b , the following holds:*

$$V'_L(\alpha(b, \sigma)) > 1. \quad (6)$$

H. *If the high-type takes a manipulative strategy at b , the following holds:*

$$V'_H(\beta(b, \sigma)) < -1. \quad (7)$$

Proof of Lemma 3: Omitted. ■

If both types take a manipulative strategy at b , then by the indifference conditions for both types the following is true:

$$[V_L(\alpha(b, \sigma)) - V_H(\alpha(b, \sigma))] - [V_L(\beta(b, \sigma)) - V_H(\beta(b, \sigma))] = 2. \quad (8)$$

We now prove that there is only one pair of bid and ask prices $\alpha(b, \sigma)$ and $\beta(b, \sigma)$ which satisfies (8).

Lemma 4. *There exists only one pair of equilibrium bid and ask prices for which both types manipulate.*

Lemma 4 states that if both types manipulate in equilibrium there is only one pair of bid and ask prices. This means that if both manipulate at the different beliefs b and b' , then the equilibrium bid and ask prices are the same. By the upper-semicontinuity of the equilibrium strategies if both manipulate at b , then both manipulate in an ϵ -neighborhood of b . Thus if both manipulate, then in an ϵ -neighborhood of b the equilibrium bid and ask prices are constant.

In the case where neither type manipulates we take the first derivative of the ask and bid prices with respect to the market maker's belief. If a low-type manipulates at b , then since the value function is strictly monotone and the slope is "steep" by the indifference condition for the low-type, we can determine the direction in which the bid and ask prices move as b decreases. More precisely, the indifference condition specifies the bid and ask prices for each belief that make the informed trader indifferent between sell and buy orders. By considering if there is another pair of bid and ask prices when b changes by a small amount we can determine in which direction the bid and ask prices change. The next Lemma summarizes this argument.

Lemma 5. *Suppose that $\sigma = (\sigma_H, \sigma_L)$ is an equilibrium strategy profile when the belief is b and $\sigma' = (\sigma'_H, \sigma'_L)$ is an equilibrium strategy profile when the belief is $b - \epsilon$. For every b and sufficiently small ϵ the followings hold:*

$$\alpha(b, \sigma) \geq \alpha(b - \epsilon, \sigma'); \quad (9)$$

and

$$\beta(b, \sigma) \geq \beta(b - \epsilon, \sigma'). \quad (10)$$

The previous lemma says that bid and ask prices are monotonically increasing. We now prove the uniqueness of equilibrium. In order to do this we consider the slope of the next-period value functions, $V'_L(\alpha(b, \sigma))$ and $V'_H(\beta(b, \sigma))$. By Lemma 3 we know that there is one critical value for the slope of each value function regarding whether or not each type manipulates. By using these values we can obtain the four cases in which the slope is greater or smaller than the critical value. By considering each case we obtain the following proposition.

Proposition 1. *The equilibrium exists uniquely.*

By Lemma 2 and Lemma 5 we can see that the current period value functions are a combination of monotonic functions in b . As a result, we can prove that (C1) holds for the current-period value functions W_H and W_L . The next proposition states this result.

Proposition 2. *The current period value function W_H is monotonically decreasing and W_L is monotonically increasing in b .*

One thing that we are missing here is the strict convexity condition (C2). This property will be discussed in detail in the next section. To end this section we remark that in the last period neither type manipulates because there is no opportunity to re-trade. As a result, we can calculate the value functions in the last period and show that they satisfy (C1) and (C2). Therefore, in the second last period there is a unique equilibrium.

Remark 1. *Suppose that $T = 3$. Equilibrium exists uniquely (without (C1) and (C2)).*

4 Characterization and Simulation

In the previous section we proved that if the value function is monotone and strictly convex in the market-maker's belief, then the equilibrium exists uniquely. In this section we will characterize the equilibrium prices under the two conditions (C1) and (C2). Moreover, we will demonstrate some simulation results of the model. Let $\sigma_H : [0, 1] \rightarrow [0, 1]$ be the high-type's equilibrium strategy of buying as a function of b and similarly let $\sigma_L : [0, 1] \rightarrow [0, 1]$ be the low-type's equilibrium strategy of buying as a function of b .

Depending on the equilibrium strategy of each type, we can classify the equilibrium into four regimes:

Regime 1: Only the low-type manipulates in equilibrium;

Regime 2: Only the high-type manipulates in equilibrium;

Regime 3: Nobody manipulates in equilibrium;

Regime 4: Both types manipulate in equilibrium.

By taking the first and second derivatives in the bid and ask prices with $\sigma_H(b) = 1$ and $\sigma_L(b) = 0$, we obtain the following result.

Proposition 3. *Suppose that nobody manipulates in equilibrium. In equilibrium, the ask price is strictly concave and the bid price is strictly convex.*

Proof of Proposition 3: Omitted. ■

By Lemma 4, we know that if Regime 4 arises in equilibrium, then there is a unique pair of bid and ask prices. The following Lemma shows that in Regime 1 and 2, the slope from the origin in equilibrium ask price is decreasing and the slope from the origin in equilibrium bid price is increasing.

Lemma 6. *In Regime 1 and 2, $\frac{\alpha(b)}{b}$ is decreasing and $\frac{\beta(b)}{b}$ is increasing in terms of b .*

The previous lemma does not say that the ask price is strictly concave and the bid price is strictly convex, although convex or concave functions satisfy the properties in Lemma 6. To see how bid and ask prices behave in equilibrium we have written a computer program. In our program we use a calibrating method called “linear interpolation.” Since in the last period of the game neither type manipulates, we can calculate the value functions in the last period as well as the bid and ask prices. We then split the interval $[0, 1]$ into n segments and linearly interpolate the value function for each type in each interval. The first case we consider is manipulation by a high-type. We do this by seeing whether or not a pair of ask and bid prices exist that make the high-type indifferent between buy and sell orders in each interval of the market makers belief. Similarly, we consider a second case where the low-type manipulates and a third case where both types manipulate. Using the bid and ask prices we obtain from this procedure, we calculate the current period value functions and repeat the procedure in the following periods. To simplify the following calculation let $f_H(b) = (1 - \mu)\gamma + \mu\sigma_H(b)$, and $f_L(b) = (1 - \mu)\gamma + \mu\sigma_L(b)$.

Consider the case where only the low-type manipulates. Expressing $f_L(b)$ as a function of the ask price $\alpha(b)$ gives:

$$f_L(b) = \frac{\alpha(b) \times (1 - b) - b(1 - \alpha(b))f_H(b)}{\alpha(b) \times (1 - b)}. \quad (11)$$

Since the high-type does not manipulate, $f_H(b) = (1 - \mu)\gamma + \mu$, which we denote by H . Then,:

$$\beta(b) = \frac{\alpha(b) \times b \times (1 - H)}{\alpha(b) - b \times H}. \quad (12)$$

We construct a new function \tilde{V}_L by a linear interpolation of V_L . Define for each $\alpha_L \in [b_k, b_{k+1}]$,

$$\tilde{V}_L(\alpha_L) = (\alpha_L - b_k) \frac{V_L(b_{k+1}) - V_L(b_k)}{(b_{k+1} - b_k)} + V_L(b_k), \quad (13)$$

and for each $\beta_L \in [b_j, b_{j+1}]$,

$$\tilde{V}_L(\beta_L) = (\beta_L - b_j) \frac{V_L(b_{j+1}) - V_L(b_j)}{(b_{j+1} - b_j)} + V_L(b_j). \quad (14)$$

Let $m_k^L = \frac{V_L(b_{k+1}) - V_L(b_k)}{b_{k+1} - b_k}$ and $m_j^L = \frac{V_L(b_{j+1}) - V_L(b_j)}{(b_{j+1} - b_j)}$. By substituting the bid price (12) into the indifference condition for the low-type and rearranging yields:

$$\begin{aligned} & \alpha_L^2 (m_k^L - 1) + \alpha_L (bH(1 - m_k^L) + b(H - 1)(1 + m_j^L) - [V_L(b_j) - V_L(b_k) + b_k m_k^L - b_j m_j^L]) \\ & + [V_L(b_j) - V_L(b_k) + b_k m_k^L - b_j m_j^L] bH = 0. \end{aligned} \quad (15)$$

Thus we obtain the following lemma.

Lemma 7. *If the low-type manipulates in equilibrium, then the equilibrium price α_L solves (15). Moreover, the equilibrium price α_L satisfies:*

$$\tilde{L} \equiv \frac{\alpha_L(1 - b) - b(1 - \alpha_L)H}{\alpha_L(1 - b)} \leq (1 - \mu)(1 - \gamma) + \mu. \quad (16)$$

Inequality (16) states that the low-type's strategy of selling cannot exceed 1. We may calculate the low-type's strategy of selling from the ask price and H . If it exceeds 1, then this strategy is not feasible. The details for Regime 2 and 4 can be found in the Appendix.

In our computer simulation we look for equilibrium pairs of bid and ask prices that satisfy the conditions for Regime 1 through 4. If there are two regimes, we ask the computer to report them. Then, we repeat this procedure recursively. The following figures describe the simulation results. Note that we have used a grid size of 100. Figures 1 to 4 show the value functions and figures 5 to 8 show the equilibrium prices.

In the figures of the bid and ask prices, there is a region of beliefs in which bid or ask prices are different between periods. Given these beliefs, manipulation arises. Since the informed trader's strategy is different between the current period and the next period and so forth, bid or ask prices are also different. The results of the simulation also show that the high-type manipulates in a region of beliefs close to 0 and the low-type manipulates in a region of beliefs close to 1. This result is somewhat counter-intuitive, because for example if the high-type manipulates in a region of beliefs close to 0, bid price will be very low and he can only obtain a little amount of money. However, motivation for manipulation is to affect the future payoffs through market maker's belief updating. Therefore, they would manipulate when the bid and ask spread is small and the slope of the next-period value function is steep.

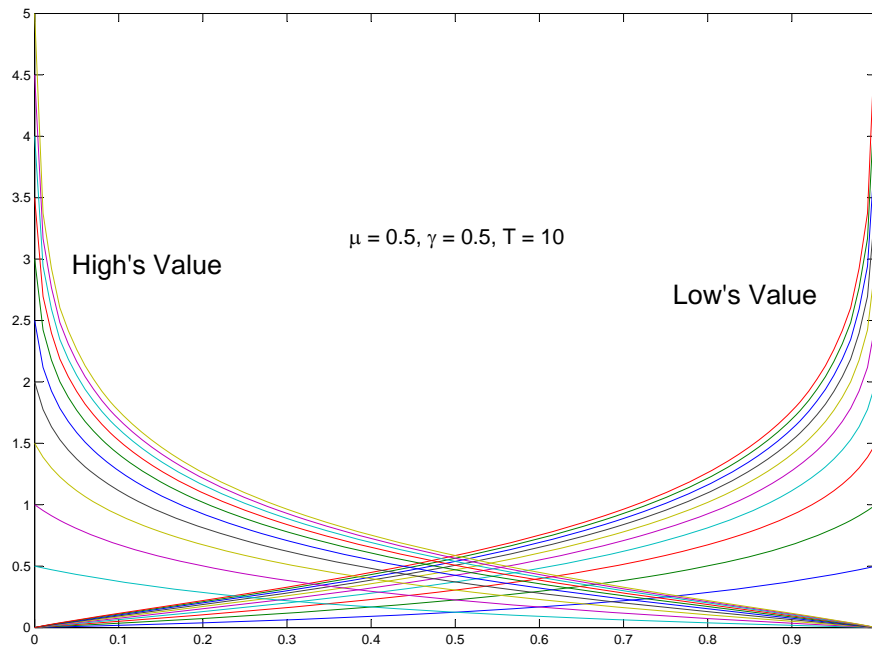


Figure 1: $\mu_0 = 0.5, \gamma = 0.5, T = 10$

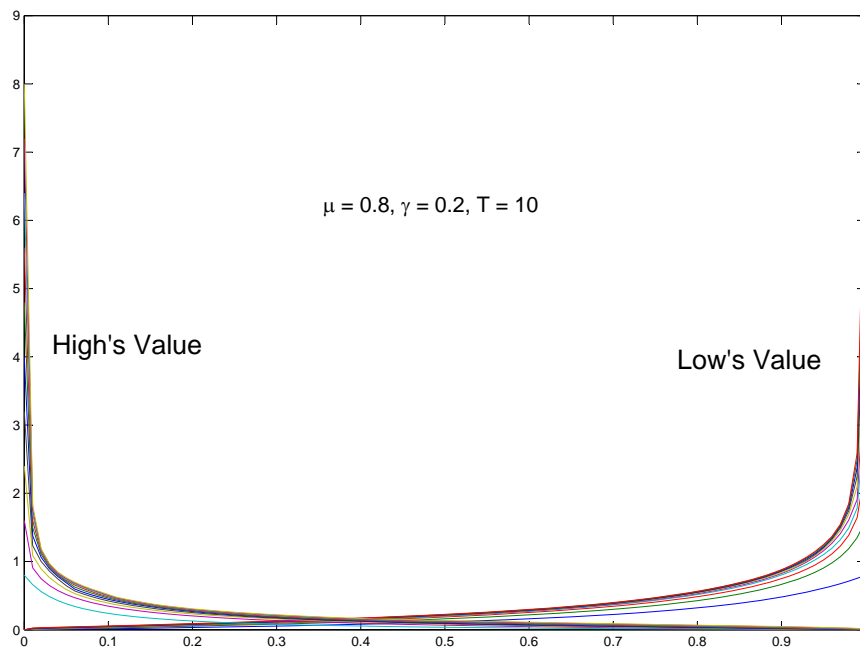


Figure 2: $\mu_0 = 0.8, \gamma = 0.2, T = 10$

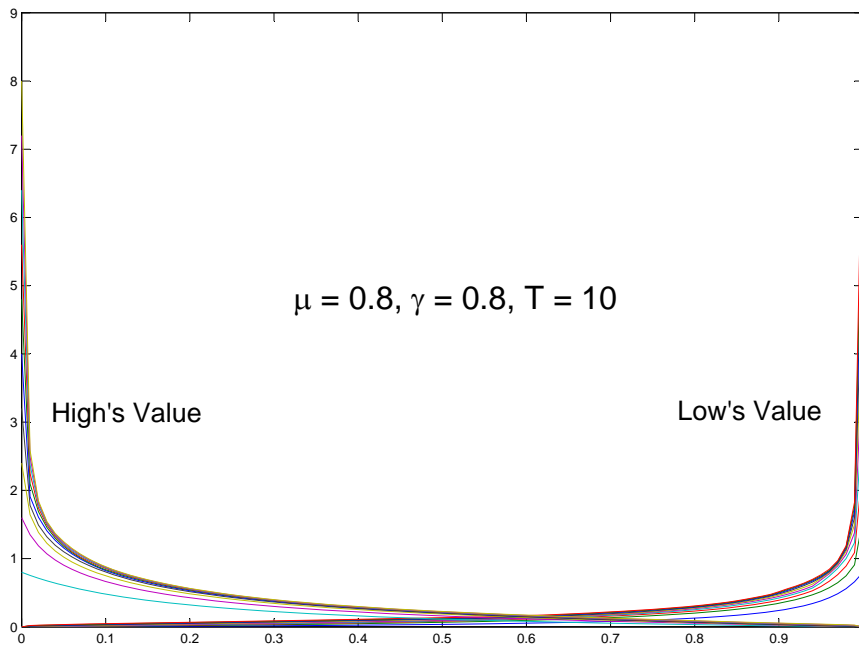


Figure 3: $\mu_0 = 0.8, \gamma = 0.8, T = 10$

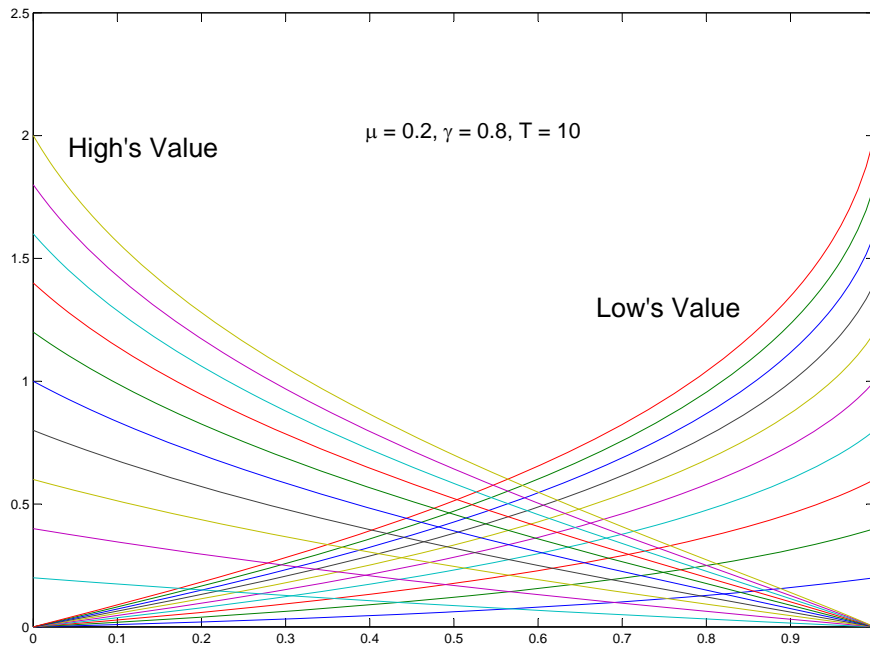


Figure 4: $\mu_0 = 0.2, \gamma = 0.8, T = 10$

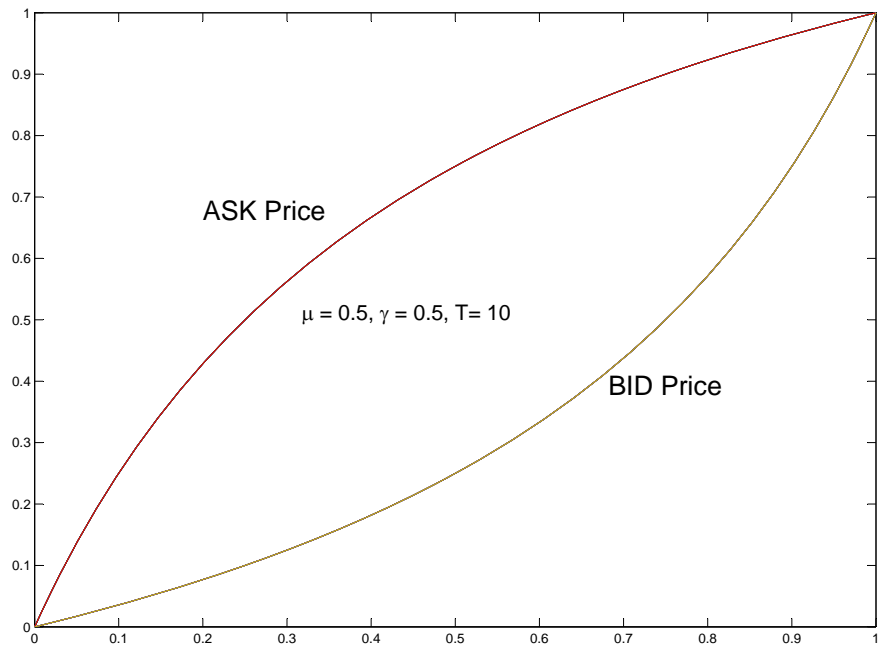


Figure 5: $\mu_0 = 0.5, \gamma = 0.5, T = 10$

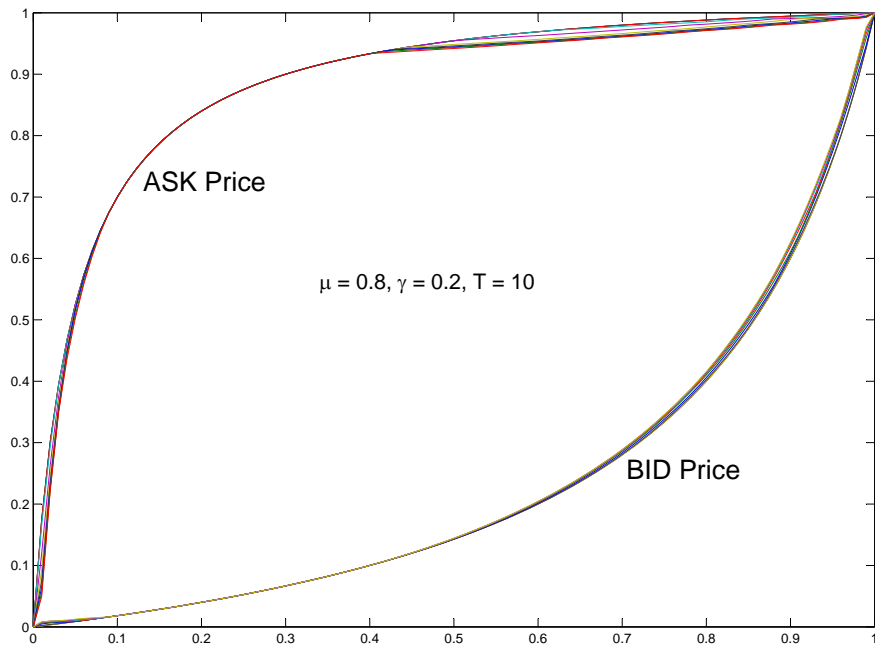


Figure 6: $\mu_0 = 0.8, \gamma = 0.2, T = 10$

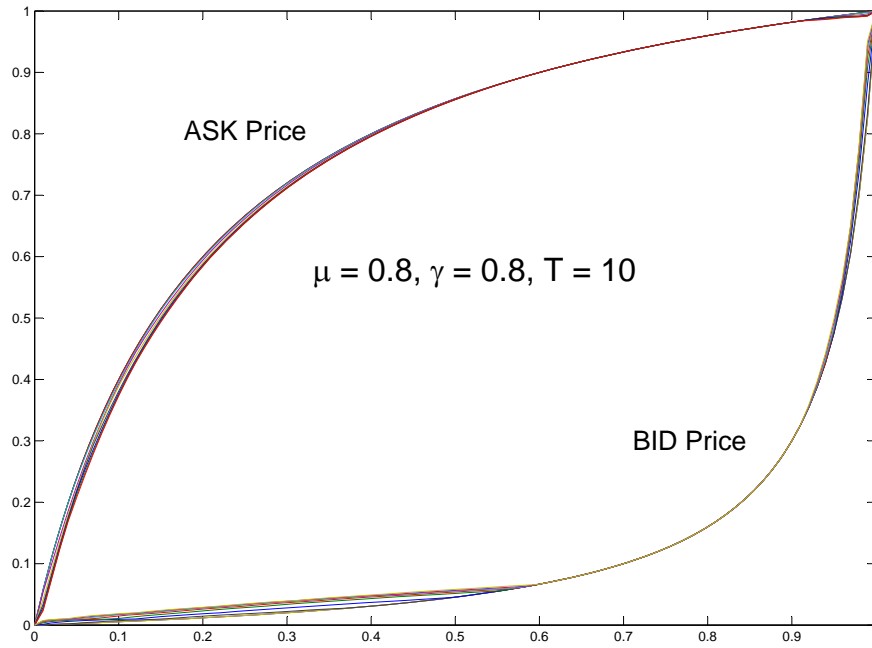


Figure 7: $\mu_0 = 0.8, \gamma = 0.8, T = 10$

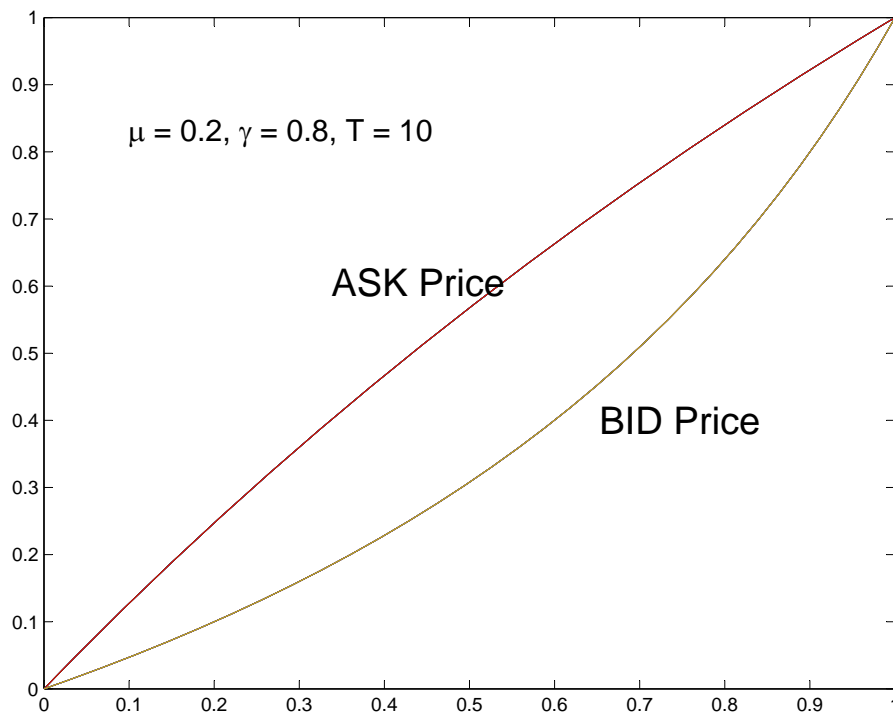


Figure 8: $\mu_0 = 0.2, \gamma = 0.8, T = 10$

Suppose that the information structure is as follows: $T = 4, \mu = 0.8, \gamma = 0.2$. Suppose that the ask price $\alpha \in [b_k, b_{k+1}]$ and the bid price $\beta \in [b_j, b_{j+1}]$. Then, by linear interpolation, we can approximate the sum of the payoffs from taking each action for each type as follows:

$$highbuy(b) = 1 - \alpha + (\alpha - b_k) \frac{V_H(b_{k+1}) - V_H(b_k)}{b_{k+1} - b_k} + V_H(b_k); \quad (17)$$

$$highsell(b) = \beta - 1 + (\beta - b_j) \frac{V_H(b_{j+1}) - V_H(b_j)}{b_{j+1} - b_j} + V_H(b_j); \quad (18)$$

$$lowbuy(b) = \alpha + (\alpha - b_k) \frac{V_L(b_{k+1}) - V_L(b_k)}{b_{k+1} - b_k} + V_L(b_k); \quad (19)$$

$$lowsell(b) = \beta + (\beta - b_j) \frac{V_L(b_{j+1}) - V_L(b_j)}{b_{j+1} - b_j} + V_L(b_j). \quad (20)$$

The table below shows some of the numerical results for $b = 0.01, 0.02, \text{ and } 0.03$. The high-type manipulates in each case.

b	0.01	0.02	0.03
$\alpha(b)$	0.1654	0.2919	0.3931
$\beta(b)$	0.0023	0.0041	0.0052
σ_{HB}	0.9309	0.9599	0.9970
$highbuy(b) = highsell(b)$	1.1058	0.8708	0.7227
$lowbuy(b)$	-0.1143	-0.1983	-0.2590
$lowsell(b)$	0.0031	0.0055	0.0071

Consider the indifference condition for the high-type. Rearranging yields:

$$2 - \alpha(b) - \beta(b) = V_H(\beta(b)) - V_H(\alpha(b)). \quad (21)$$

By Lemma 3 we know that if manipulation arises in some region of a belief, then the value function V_H needs to be steep enough in that region. In other words, manipulation changes the market maker's posterior belief from $\alpha(b)$ to $\beta(b)$. Consequently, if manipulation arises, then there is a difference between $V_H(\alpha(b))$ and $V_H(\beta(b))$. However, if this difference is too large, then (21) does not hold. This effect can be seen in the results of our numerical simulation which show that manipulation arises in the region of beliefs where the value function is steep and the bid and ask spread is small.

From our numerical simulation we make the following conjectures. First, there is an interval of beliefs in which each type manipulates. Our first conjecture is that both types do not manipulate at the same time. Our second conjecture is that as the time period increases, manipulation occurs at wider ranges of the market maker's belief. In the previous example manipulation arises between $b = 0.1$ to 0.4 for $t = 5$, $b = 0.1$ to 0.5 for $t = 6$ and 7^1 . Our last, and most important conjecture, is that the value functions are strictly convex, the ask price is strictly concave and the bid price is strictly convex. If the

¹We ran many other examples and this conjecture was observed in all cases.

third conjecture is proved, it would complete the proof of uniqueness result in a general case. To do so, proving the first conjecture would simplify the proof for a general case, because in this way we can focus on one type.

5 Remarks

Notice that in the last period the informed trader trades on their information because there is no chance to re-trade. This means that in the last period of the game the informed trader's unique equilibrium strategy is to trade honestly. By taking the first and second derivative of the last-period value function we can see that the value of the game in the last period is strictly monotonic and convex in the market maker's belief. The idea behind proving the uniqueness of equilibrium in the general case is to prove the uniqueness of the equilibrium strategy, supposing the existence of unique next-period value functions V_L and V_H , which are monotonic and convex in terms of the market maker's belief.

From the results of our numerical simulation we make several conjectures. First, in equilibrium both types do not manipulate at the same time. Second, the bid and ask price is strictly monotonic and the bid price is strictly convex. Third, the value functions are monotonic and strictly convex. These conjectures are important steps towards a proof of the uniqueness of equilibrium in the general case. One specific difficulty in proving uniqueness for the general case can be seen by considering the indifference condition for the low-type. If the low-type manipulates the following must hold:

$$-\alpha(b) + V_L(\alpha(b)) = \beta(b) + V_L(\beta(b)). \quad (22)$$

If the equilibrium is unique, then the equilibrium strategy should be continuous. Therefore, if the low-type manipulates at belief b then he should manipulate in an ϵ -neighborhood of b . This means that in an ϵ -neighborhood of b , a similar indifference condition must hold. The question comes down to the properties of V_L , α and β that make the two indifference conditions hold in an ϵ -neighborhood of b . Ultimately, the question becomes how the low-type changes his strategy in an ϵ -neighborhood of b . Similar questions arise if we consider the indifference condition for the high type. Regime 4 is the most difficult case because we have to consider how both types change their strategies in an ϵ -neighborhood of b . This is why proving our first conjecture that both types do not manipulate at the same time is a key step in proving uniqueness in the general case.

The last difficulty is to prove that the value functions are strictly convex in the market maker's prior belief. If V_L is monotone and strictly convex we can write the current-period value function as:

$$W_L(b) = \mu (\beta(b) + V_L(\beta(b))) + (1 - \mu) (\gamma V_L(\alpha(b)) + (1 - \gamma) V_L(\beta(b))). \quad (23)$$

Notice that if β and V_L are strictly convex and monotone, then $V_L(\beta(b))$ is strictly convex. The problem is the ask price. If α is strictly concave, then we cannot determine if $V_L(\alpha(b))$ is strictly concave. This prevents us from determining if W_L is strictly convex.

Proving the uniqueness of equilibrium in the general case is a challenging endeavor and this paper opens up the path to it. In this paper we have presented a model of dynamic informed trading, in which there exists a unique equilibrium under two conditions for the value functions. We have presented a computational method to solve for an equilibrium and have made several conjectures for proving the uniqueness of equilibrium in the general case.

Appendix: Proof of Theorem 1

In order to prove the existence of equilibrium, we consider the equilibrium strategies (σ_L^*, σ_H^*) to be a fixed point of the collection of their best response correspondences $BR = \{BR^t\}_{t=1, \dots, T}$ with $BR^t : [\Delta(A)]^2 \Rightarrow [\Delta(A)]^2$ such that for each t , $(\sigma_L^*, \sigma_H^*) = BR^t(\sigma_L^*, \sigma_H^*)$. Let $U_n^t : \Delta(A) \times [0, 1]^2 \rightarrow \mathbb{R}$ denote the payoff function for the type $n \in N$ trader in period t . More formally, for $n \in \{H, L\}$,

$$U_n^t(\sigma_n, p_t) = \sum_{t'=t}^T [\sigma_{nB}(\theta - \alpha_{t'}) - \sigma_{nS}(\theta - \beta_{t'})]. \quad (24)$$

Then, we define the informed trader's best response correspondence: for every $t \in \{1, \dots, T\}$ and given p_t ,

$$BR^t(\sigma_L, \sigma_H) = \left\{ (\sigma_L, \sigma_H) \in [\Delta(A)]^2 \mid \sigma_n \in \arg \max_{\sigma \in \Delta(A)} U_n^t(\sigma, p_t) \quad \forall n \in N \right\}. \quad (25)$$

Therefore, when $b(h_t) = b_t$, $\alpha_t^*(b(h_t)) = \alpha_t$ and $\beta_t^*(b(h_t)) = \beta_t$, continuation value of the game for the high-type in period t is:

$$V_H^t(b_t) = \max_{\sigma_H \in \Delta(A)} [\sigma_{HB}(1 - \alpha_t + V_H^{t+1}(b(h_t), B)) + \sigma_{HS}(\beta_t - 1 + V_H^{t+1}(b(h_t), S))], \quad (26)$$

and one for the low type is:

$$V_L^t(b_t) = \max_{\sigma_L \in \Delta(A)} [-\sigma_{LB}\alpha_t + V_L^{t+1}(b(h_t), B) + \sigma_{LS}(\beta_t + V_L^{t+1}(b(h_t), S))]. \quad (27)$$

Thus, an equilibrium defined in Definition 3 is a fixed point of the best response correspondence BR , and α_t and β_t are respectively updated by Bayes rule.

Lemma 8. *The payoff function U_n^t is continuous. In addition, for every t , BR^t is a upper semi-continuous correspondence.*

Proof: Since the argument is symmetric, we only consider the high-type's payoff function and the value function. Note that U_H^t is continuous in his strategy and also the market maker's quotes (β_t, α_t) . Then, U_H^t is a continuous numerical function.

We respectively denote the sequences of prices associated with σ^k and $\hat{\sigma}^k$ by p^k and \hat{p}^k and also σ and $\hat{\sigma}$ by p and \hat{p} . Then, since the prices are continuous in strategies, we have $p^k \rightarrow p$ and $\hat{p}^k \rightarrow \hat{p}$.

Now on the contrary, suppose that there exists a sequence as above but $\hat{\sigma} \notin BR(\sigma_H, \sigma_L)$. Without loss of generality, we suppose that there exists a $\epsilon > 0$ and $\bar{\sigma}_H \in \Delta(E)$ such that:

$$U_H^t(\bar{\sigma}_H, p) > U_H^t(\hat{\sigma}_H, p) + 3\epsilon. \quad (28)$$

For k large enough, by continuity of the payoff function and prices, we have:

$$\begin{aligned} U_H^t(\bar{\sigma}_H, p^k) &> U_H^t(\bar{\sigma}_H, p) - \epsilon > U_H^t(\hat{\sigma}_H, p) + 2\epsilon \\ &> U_H^t(\hat{\sigma}_H^k, p) + \epsilon > U_H^t(\hat{\sigma}_H^k, p^k). \end{aligned} \quad (29)$$

This contradicts with the fact that $(\hat{\sigma}_H^k, \hat{\sigma}_L^k) \in BR^t(\sigma_H^k, \sigma_L^k)$ for all k . ■

Lemma 9. *The set $[\Delta(A)]^2$ is non-empty, compact and convex.*

Proof: The set of strategies $\Delta(A)$ is non-empty, compact and convex. The set $[\Delta(A)]^2$ is a Cartesian product of those sets and thus the result follows. ■

Lemma 10. *The informed trader's best response correspondence BR^t is non-empty and convex-valued for every $t \in \{1, \dots, T\}$.*

Proof: We will prove this by mathematical induction. Since the argument is symmetric, we only consider the high type. Consider the last period $t = T$. Then, the high type and low type trade on their information. In this sense, BR^T is non-empty and convex-valued. Next we suppose that in period $t+1$, BR^{t+1} is non-empty and convex-valued. Then, we will prove that in period t , BR^t is also non-empty and convex-valued.

By the assumption for the inductive hypothesis, we know that V_H^{t+1} is well-defined. Now, fix a history h^{t-1} arbitrarily. Then, given V_H^{t+1} , the right hand side of the expression in (26) is linear in the strategies σ_H . Therefore the expression in (26) has a maximum so that the set BR^t is non-empty.

Second, we will prove that it is also convex-valued. Take two different strategies $(\bar{\sigma}_H, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$ and $(\bar{\bar{\sigma}}_H, \bar{\bar{\sigma}}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. We denote the prices associated with the strategies $(\bar{\sigma}_H, \bar{\sigma}_L)$ by \bar{p}_t . Then, the following must hold: $U_H^t(\bar{\sigma}_H, \bar{p}_t) = U_H^t(\bar{\bar{\sigma}}_H, \bar{p}_t)$.

Let $\hat{\sigma}_H^t = \gamma\bar{\sigma}_H + (1 - \gamma)\bar{\bar{\sigma}}_H$ for some $\gamma \in (0, 1)$. By using linearity of the payoff function, we conclude: $(\hat{\sigma}_H^t, \bar{\sigma}_L) \in BR^t(\bar{\sigma}_H, \bar{\sigma}_L)$. Therefore, BR^t is convex-valued. ■

Proof of Theorem 1: By Lemma 8 to Lemma 10, we can apply the Kakutani's fixed point theorem to the best response correspondence BR^t on $[\Delta(A)]^2$ for all $t \in \{1, \dots, T\}$. ■

Appendix: Proofs

Proof of Lemma 1 - 1: On the contrary, suppose that for some b , bid-ask spread is negative. That is,

$\alpha(b, \sigma) \leq \beta(b, \sigma)$. Then, we have:

$$1 - \alpha(b, \sigma) + V_H(\alpha(b, \sigma)) > \beta(b, \sigma) - 1 + V_H(\beta(b, \sigma)); \quad (30)$$

$$-\alpha(b, \sigma) + V_L(\alpha(b, \sigma)) < \beta(b, \sigma) + V_L(\beta(b, \sigma)). \quad (31)$$

Then, it must be the case that in equilibrium $\sigma_{HB} = 1$ and $\sigma_{LB} = 0$. Then, by the Bayes rule, we have:

$$\alpha(b, \sigma) > b > \beta(b, \sigma), \quad (32)$$

which contradicts with our assumption. \square

Proof of Lemma 1 - 2: The result follows from 1 and Bayes rule. \square \blacksquare

Proof of Lemma 4: We define $J(\alpha, \beta) \equiv \begin{pmatrix} H(\alpha, \beta) \\ L(\alpha, \beta) \end{pmatrix}$ where for $\alpha \in [0, 1]$ and $\beta \in [0, 1]$,

$$H(\alpha, \beta) = V_H(\alpha) - V_H(\beta) + 2 - \alpha - \beta, \quad (33)$$

and

$$L(\alpha, \beta) = V_L(\alpha) - V_L(\beta) - \alpha - \beta. \quad (34)$$

Since V_H is a decreasing function and V_L is an increasing function and by Lemma 3, the determinant of J denoted by \mathcal{J} satisfies:

$$\mathcal{J}(\alpha, \beta) = -[V_H'(\alpha) - 1] \times [V_L'(\beta) + 1] + [V_H'(\beta) + 1] \times [V_L'(\alpha) - 1] > 0. \quad (35)$$

Since the elements in the upper left corner of \mathcal{J} and the lower right corner of \mathcal{J} are both strictly negative, we conclude that \mathcal{J} is negative definite. Take two distinct $p_1 = (\alpha_1, \beta_1)$ and $p_2 = (\alpha_2, \beta_2)$. Therefore, we have: $\mathcal{J}(p_1) \neq \mathcal{J}(p_2)$. Finally, we conclude that there exists only one pair of α and β which satisfies: $H(\alpha, \beta) = 0$ and $L(\alpha, \beta) = 0$. \blacksquare

Proof of Lemma 5: When nobody manipulates, by the Bayes rule we can show that bid and ask prices decrease as market makers' belief b decreases. By Lemma 4, there are no two distinct pairs of bid and ask prices at which both types manipulate. Now, we suppose that only one type manipulates. Since the argument is symmetric, suppose that the low-type manipulates at b . Then, we have:

$$-\alpha(b, \sigma) + V_L(\alpha(b, \sigma)) = \beta(b, \sigma) + V_L(\beta(b, \sigma)); \quad (36)$$

$$-\alpha(b - \epsilon, \sigma') + V_L(\alpha(b - \epsilon, \sigma')) \leq \beta(b - \epsilon, \sigma') + V_L(\beta(b - \epsilon, \sigma')). \quad (37)$$

Then, we obtain:

$$\begin{aligned} & (\alpha(b, \sigma) - \alpha(b - \epsilon, \sigma')) \left(\frac{V_L(\alpha(b - \epsilon, \sigma')) - V_L(\alpha(b, \sigma))}{\alpha(b - \epsilon, \sigma') - \alpha(b, \sigma)} - 1 \right) \\ & \geq (\beta(b, \sigma) - \beta(b - \epsilon, \sigma')) \left(1 + \frac{V_L(\beta(b, \sigma)) - V_L(\beta(b - \epsilon, \sigma'))}{\beta(b, \sigma) - \beta(b - \epsilon, \sigma')} \right). \end{aligned} \quad (38)$$

By Lemma 3 and Lemma 4, we can see that bid and ask prices move in the same direction, which is impossible when only the low-type manipulates. ■

Lemma 11. *Suppose that in equilibrium one type takes a manipulative strategy and the other type does not. Then, the equilibrium strategy exists uniquely.*

Proof of Lemma 11: Since the argument is symmetric, we will prove the result for the case where only the high-type manipulates. We will show that given b and σ_L there is a unique pair of strategies σ_H satisfying the condition that the high-type is indifferent between buy and sell. On the contrary, suppose that there are different strategies $\hat{\sigma}_H$ satisfying the indifference condition for the high-type with the prices $\alpha(b, \hat{\sigma})$ and $\beta(b, \hat{\sigma})$. Now, suppose that: $\sigma_{HB} < \hat{\sigma}_{HB} \leq 1$. Then, we have: $\alpha(b, \hat{\sigma}) > \alpha(b, \sigma)$ and $\beta(b, \hat{\sigma}) < \beta(b, \sigma)$. Then, we have:

$$1 - \alpha(b, \hat{\sigma}) + V_H(\alpha(b, \hat{\sigma})) \geq \beta(b, \hat{\sigma}) - 1 + V_H(\beta(b, \hat{\sigma})). \quad (39)$$

Thus, we have:

$$\alpha(b, \hat{\sigma}) - \alpha(b, \sigma) - V_H(\alpha(b, \hat{\sigma})) + V_H(\alpha(b, \sigma)) \leq \beta(b, \sigma) - \beta(b, \hat{\sigma}) + V_H(\beta(b, \sigma)) - V_H(\beta(b, \hat{\sigma})). \quad (40)$$

By Lemma 3, we have:

$$\beta(b, \sigma) - \beta(b, \hat{\sigma}) + V_H(\beta(b, \sigma)) - V_H(\beta(b, \hat{\sigma})) < 0. \quad (41)$$

However, since V_H is decreasing, the left hand side of (40) is strictly greater than 0, which makes (40) impossible to hold. ■

Lemma 12. *If the equilibrium bid and ask prices are unique at b , the equilibrium strategies are unique.*

Proof of Lemma 12: For the simplicity of notation, let $f_H = (1 - \mu)\gamma + \mu\sigma_{HB}$ and $f_L = (1 - \mu)\gamma + \mu\sigma_{LB}$. Suppose that in equilibrium, there are two different pairs of strategies, σ and $\hat{\sigma}$. Now on the contrary, suppose that $\alpha(b, \hat{\sigma}) = \alpha(b, \sigma)$ and $\beta(b, \hat{\sigma}) = \beta(b, \sigma)$. Similarly with f_H and f_L , we define \hat{f}_H and \hat{f}_L associated with $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{HB}$. By the Bayes rule, we can write: $\alpha(b, \sigma) = \frac{f_H b}{f_H b + (1-b)f_L}$, and $\alpha(b, \hat{\sigma}) = \frac{\hat{f}_H b}{\hat{f}_H b + (1-b)\hat{f}_L}$. Since $\alpha(b, \hat{\sigma}) = \alpha(b, \sigma)$, we must have:

$$\hat{f}_H f_L = \hat{f}_L f_H. \quad (42)$$

Similarly, we have $\beta(b, \sigma) = \frac{(1-f_H)b}{(1-f_H)b + (1-b)(1-f_L)}$, and $\beta(b, \hat{\sigma}) = \frac{(1-\hat{f}_H)b}{(1-\hat{f}_H)b + (1-b)(1-\hat{f}_L)}$. By equating them, we must have:

$$(1 - \hat{f}_H)(1 - f_L) = (1 - \hat{f}_L)(1 - f_H). \quad (43)$$

Combining the equations (42) and (43) gives $\hat{f}_H - f_H = f_L - \hat{f}_L \equiv \Delta$. Then, by substituting it into (42) we obtain:

$$(f_H + \Delta)f_L = (f_L + \Delta)f_H. \quad (44)$$

Therefore, we must have $f_H = f_L$ and $\hat{f}_H = \hat{f}_L$. Conversely, if $f_H = f_L$ and $\hat{f}_H = \hat{f}_L$, then $\beta(b, \hat{\sigma}) = \beta(b, \sigma) = b$ and $\alpha(b, \hat{\sigma}) = \alpha(b, \sigma) = b$. This contradicts with Lemma 1. ■

Proof of Proposition 1:

Case 1: $V'_L(\alpha(b, \sigma)) > 1$ and $V'_H(\beta(b, \sigma)) < -1$

In this case, both types could either manipulate or not manipulate in equilibrium. By Lemma 11, if one of the two types manipulates and the other does not, there exists a unique equilibrium. It remains to show that one equilibrium in which both do not manipulate does not co-exist with the other equilibrium in which both manipulate.

Call a non-manipulative equilibrium strategy σ and a manipulative equilibrium strategy $\hat{\sigma}$. Then, we have: $\alpha(b, \hat{\sigma}) < \alpha(b, \sigma)$ and $\beta(b, \hat{\sigma}) > \beta(b, \sigma)$.

Then, for the high-type, we have:

$$1 - \alpha(b, \hat{\sigma}) + V_H(\alpha(b, \hat{\sigma})) = \beta(b, \hat{\sigma}) - 1 + V_H(\beta(b, \hat{\sigma})), \quad (45)$$

and

$$1 - \alpha(b, \sigma) + V_H(\alpha(b, \sigma)) > \beta(b, \sigma) - 1 + V_H(\beta(b, \sigma)). \quad (46)$$

Similarly, for the low-type, we have:

$$-\alpha(b, \hat{\sigma}) + V_L(\alpha(b, \hat{\sigma})) = \beta(b, \hat{\sigma}) + V_L(\beta(b, \hat{\sigma})), \quad (47)$$

and

$$-\alpha(b, \sigma) + V_L(\alpha(b, \sigma)) < \beta(b, \sigma) + V_L(\beta(b, \sigma)). \quad (48)$$

Thus, we must have:

$$\begin{aligned} & V_L(\beta(b, \sigma)) - V_L(\beta(b, \hat{\sigma})) - [V_L(\alpha(b, \sigma)) - V_L(\alpha(b, \hat{\sigma}))] \\ & > -\alpha(b, \sigma) + \alpha(b, \hat{\sigma}) - [\beta(b, \sigma) - \beta(b, \hat{\sigma})] \\ & > V_H(\beta(b, \sigma)) - V_H(\beta(b, \hat{\sigma})) - [V_H(\alpha(b, \sigma)) - V_H(\alpha(b, \hat{\sigma}))]. \end{aligned} \quad (49)$$

Since $\alpha(b, \hat{\sigma}) < \alpha(b, \sigma)$, $\beta(b, \hat{\sigma}) > \beta(b, \sigma)$, V_H is decreasing and V_L is increasing, the inequality (49) is impossible. This complete our proof in this case. □

Case 2: $V'_L(\alpha(b, \sigma)) > 1$ and $V'_H(\beta(b, \sigma)) \geq -1$

In this case, the high-type does not manipulate. The low-type either manipulate or does not. The proof is done by Lemma 11. □

Case 3: $V'_L(\alpha(b, \sigma)) \leq 1$ and $V'_H(\beta(b, \sigma)) < -1$

In this case, the high-type does not manipulate. The low-type either manipulate or does not. Similarly with Case 2, the proof is done by Lemma 11. □

Case 4: $V'_L(\alpha(b, \sigma)) \leq 1$ and $V'_H(\beta(b, \sigma)) \geq -1$

In this case, both types do not manipulate and therefore, the equilibrium exists uniquely. □

In the end, we conclude that equilibrium exists uniquely in every case. ■

Proof of Proposition 2:

· **When $t = T$**

Since this is the last chance to trade, both types trade on their information. Therefore,

$$W_H(b) = \left(1 - \frac{[\mu + (1 - \mu)\gamma]b}{(1 - \mu)\gamma + \mu b}\right) \cdot \mu = \frac{(1 - b)(1 - \mu)\gamma}{(1 - \mu)\gamma + \mu b} \cdot \mu. \quad (50)$$

Therefore, we conclude that W_H is strictly decreasing in b . □

· **When $t = 1, \dots, T - 1$**

By Proposition 1, equilibrium exists uniquely. Let $b > b'$, and σ' denotes the equilibrium strategy when the belief is b' . Then, we have:

$$\begin{aligned} W_H(b) &= \mu(1 - \alpha(b, \sigma) + V_H(\alpha(b, \sigma))) + (1 - \mu)(\gamma V_H(\alpha(b, \sigma)) + (1 - \gamma)V_H(\beta(b, \sigma))) \\ &< \mu(1 - \alpha(b', \sigma') + V_H(\alpha(b', \sigma'))) + (1 - \mu)(\gamma V_H(\alpha(b', \sigma')) + (1 - \gamma)V_H(\beta(b', \sigma'))) \\ &= W_H(b'). \end{aligned}$$

This completes our proof. ■

Let: $P(b) = f_H(b) \times b + f_L(b) \times (1 - b)$.

Lemma 13. *Fix a history h^t arbitrarily and suppose that $b = b(h^t)$. Suppose that V_H is monotonically decreasing and strictly convex in the market maker's prior b and that V_L is monotonically increasing and strictly convex in b . Suppose that only one type manipulates at the belief b . Then, P is increasing at the belief b .*

Proof of Lemma 13:

Case 1: When the high-type manipulates at the belief b .

By the indifference condition, we know that if the high-type manipulates, then the slope between bid and ask prices should be:

$$d_H(b) = \frac{\alpha(b) + \beta(b) - 2}{\alpha(b) - \beta(b)}. \quad (51)$$

Thus, we obtain:

$$d'_H(b) = \frac{\alpha'(b)(1 - \beta(b)) - \beta'(b)(1 - \alpha(b))}{[\alpha(b) - \beta(b)]^2} \times 2. \quad (52)$$

Suppose that the high-type manipulates at the belief b . Then by the continuity of the equilibrium strategy, we must have $d'_H(b) > 0$ because the equilibrium strategy is unique. By the Bayes rule, we have:

$$\log(1 - \alpha(b)) = \log(1 - b) + \log(1 - \mu)\gamma - \log P(b), \quad (53)$$

and

$$\log(1 - \beta(b)) = \log(1 - b) + \log((1 - \mu)(1 - \gamma) + \mu) - \log(1 - P(b)). \quad (54)$$

By taking the first derivative of the above equations, we obtain:

$$\frac{-\alpha'(b)}{1 - \alpha(b)} = \frac{-1}{1 - b} - \frac{P'(b)}{P(b)}, \quad (55)$$

and

$$\frac{-\beta'(b)}{1 - \beta(b)} = \frac{-1}{1 - b} + \frac{P'(b)}{1 - P(b)}. \quad (56)$$

Notice that:

$$\alpha'(b)(1 - \beta(b)) - \beta'(b)(1 - \alpha(b)) = (1 - \alpha(b))(1 - \beta(b)) \times P'(b) \times \left(\frac{1}{P(b)} + \frac{1}{1 - P(b)} \right).$$

Therefore, we have:

$$d'_H(b) = \frac{(1 - \alpha(b))(1 - \beta(b)) \times P'(b) \times \left(\frac{1}{P(b)} + \frac{1}{1 - P(b)} \right)}{[\alpha(b) - \beta(b)]^2} \times 2. \quad (57)$$

and thus we conclude $P'(b) > 0$. \square

Case 2: When the low-type manipulates at the belief b .

By the indifference condition, we know that if the low-type manipulates, then the slope between bid and ask prices should be:

$$d_L(b) = \frac{\alpha(b) + \beta(b)}{\alpha(b) - \beta(b)}. \quad (58)$$

Thus, we obtain:

$$d'_L(b) = \frac{\alpha(b)\beta'(b) - \alpha'(b)\beta(b)}{[\alpha(b) - \beta(b)]^2} \times 2. \quad (59)$$

Suppose that the low-type manipulates at the belief b . Then by the continuity of the equilibrium strategy, we must have $d'_L(b) > 0$ because the equilibrium strategy is unique. By the Bayes rule, we have:

$$\log \alpha(b) = \log b + \log((1 - \mu)\gamma + \mu) - \log P(b), \quad (60)$$

and

$$\log \beta(b) = \log b + \log(1 - \mu)(1 - \gamma) - \log(1 - P(b)). \quad (61)$$

By taking the first derivative of the above equations, we obtain:

$$\frac{\alpha'(b)}{\alpha(b)} = \frac{1}{b} - \frac{P'(b)}{P(b)}, \quad (62)$$

and

$$\frac{\beta'(b)}{\beta(b)} = \frac{1}{b} + \frac{P'(b)}{1 - P(b)}. \quad (63)$$

Notice that:

$$\alpha(b)\beta'(b) - \alpha'(b)\beta(b) = \alpha(b)\beta(b) \times P'(b) \times \left(\frac{1}{1-P(b)} + \frac{1}{P(b)} \right). \quad (64)$$

Thus, we can write:

$$d'_L(b) = \frac{\alpha(b)\beta(b) \times P'(b) \times \left(\frac{1}{1-P(b)} + \frac{1}{P(b)} \right)}{[\alpha(b) - \beta(b)]^2} \times 2. \quad (65)$$

Thus, we must have $P'(b) > 0$. ■

Proof of Lemma 6:

Regime 1: By the Bayes rule, we have:

$$b \times ((1 - \mu)\gamma + \mu) = P(b) \times \alpha(b), \quad (66)$$

and

$$b(1 - \mu)(1 - \gamma) = (1 - P(b)) \times \beta(b). \quad (67)$$

By Lemma 13, for $k < 1$, we have:

$$\frac{P(b)}{P(kb)} = \frac{\alpha(kb)}{k\alpha(b)} > 1, \quad (68)$$

and

$$\frac{1 - P(b)}{1 - P(kb)} = \frac{\beta(kb)}{k\beta(b)} < 1. \quad (69)$$

Dividing both sides by b and arranging terms yields the desired results. □

Regime 2: The proof will be done in a similar fashion with the previous lemma. □ ■

Appendix: Numerical Simulation

Let $m_k^H = \frac{V_H(b_{k+1}) - V_H(b_k)}{b_{k+1} - b_k}$ and $m_j^H = \frac{V_H(b_{j+1}) - V_H(b_j)}{(b_{j+1} - b_j)}$. Moreover, let $(1 - \mu)(1 - \gamma) + \mu = L$.

Lemma 14. *When the high-type manipulates, then equilibrium ask price α_H solves:*

$$\begin{aligned} & (m_k^H - 1)\alpha_H^2 + \alpha_H (-b_k m_k^H + 1 + V_H(b_k) + (b + L(1 - b))(1 - m_k^H)) \\ & + \alpha_H [L(1 - b) + [b_j - 1 + L(1 - b)] m_j^H - V_H(b_j)] \\ & - (-b_k m_k^H + 1 + V_H(b_k))(b + L(1 - b)) \\ & + (-L(1 - b) + [b(1 - b_j) - b_j L(1 - b)] m_j^H + V_H(b_j) (b + L(1 - b))) = 0. \end{aligned} \quad (70)$$

Moreover, equilibrium ask price α_H must satisfy:

$$\tilde{H} \equiv \frac{\alpha(1 - L)(1 - b)}{b(1 - \alpha)} \leq (1 - \mu)\gamma + \mu. \quad (71)$$

Proof of Lemma 14: To avoid lengthy calculations, we only explain the key steps. Then, the equation in question is:

$$1 - \alpha + (\alpha - b_k)m_k^H + V_H(b_k) = \frac{(b - \alpha) + \alpha L(1 - b)}{(b - \alpha) + L(1 - b)} - 1 + \left(\frac{(b - \alpha) + \alpha L(1 - b)}{(b - \alpha) + L(1 - b)} - b_j \right) m_j^H + V_H(b_j).$$

It is very similarly done with Lemma 7. Finally, we obtain the desired result. ■

$$\text{Let } m_k^D = \frac{D(b_{k+1}) - D(b_k)}{b_{k+1} - b_k} \text{ and } m_j^D = \frac{D(b_{j+1}) - D(b_j)}{b_{j+1} - b_j}.$$

Lemma 15. *If both types manipulate, then the equilibrium ask price α_M and strategy σ_{HB}^* satisfies:*

$$\begin{aligned} m_k^D \alpha_M^2 + \alpha_M [-b_k m_k^D + D(b_k) - bH^* m_k^D + (b(H^* - 1) + b_j) m_j^D - (D(b_j) + 2)] \\ + bH^*(D(b_j) + 2 - b_j m_j^D + b_k m_k^D - D(b_k)) = 0, \end{aligned} \quad (72)$$

and

$$H^* = (1 - \mu)\gamma + \mu\sigma_{HB}^*. \quad (73)$$

Proof of Lemma 15: For the simplicity of notation, let us define: $D = V_L - V_H$. If both types manipulate, this means that there is a pair of ask and bid prices which satisfies: $D(\alpha) = D(\beta) + 2$. By using the linear interpolating method, we check if there is a pair of α , b and H which satisfies the indifference conditions for both types:

$$(\alpha - b_k)m_k^D + D(b_k) = \left[\frac{\alpha b(H - 1)}{(bH - \alpha)} - b_j \right] m_j^D + D(b_j) + 2. \quad (74)$$

Then, we obtain the desired result. ■

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